Finite Frames

P. Sam Johnson

NITK, Surathkal, India

Introduction to Finite Frames

One of the important concepts in the study of vector spaces is the concept of a basis for the vector space, which allows every vector to be uniquely represented as a linear combination of the basis elements.

However, the linear independence property for a basis is restrictive; sometimes it is impossible to find vectors which both fulfill the basis requirements and also satisfy external conditions (for example, orthonormal basis condition, Parseval's identity and so on) demanded by applied problems. For such purposes, we need to look for more flexible types of spanning sets. **Frames provide these alternatives.**

In this lecture, we discuss frames in $\mathbb{R}^n$, giving a nonstandard but equivalent definition of a frame. We use this as a first definition in hopes that it lends some intuition about frames that carry over into the more general setting.

We then proceed to give the standard definition of a frame in a finite dimensional Hilbert space.

We begin our discussion of frames by restricting ourselves to real Euclidean space $\mathbb{R}^n$ with the inner product $\langle x, y \rangle = y^* x$. The definition we give for a frame in this setting differs from, but is equivalent to, the definition of a more general frame.

Definition 1.

Let $k \geq n$ and let $\{v_1, v_2, \ldots, v_k\}$ be a finite sequence of vectors in $\mathbb{R}^n$. We say that **the sequence $\{v_1, v_2, \ldots, v_k\}$ has an extension to a basis for $\mathbb{R}^k$** *if there exist vectors $\{w_1, w_2, \ldots, w_k\}$ in $\mathbb{R}^{k-n}$ such that*

$$\left\{ \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ w_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_k \\ w_k \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^k$.

Definition 2.

*An $\mathbb{R}^n$-**frame** is a finite sequence of vectors $F = \{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$, with $k \geq n$, such that there exists an extension of F to a basis for $\mathbb{R}^k$.*

It is clear from the definition that the $\mathbb{R}^n$-frames having exactly n elements (so $k = n$) are precisely the bases for $\mathbb{R}^n$.

Lemma 3.

A collection of vectors in $\mathbb{R}^n$ is an $\mathbb{R}^n$-frame if and only if it is a spanning set for $\mathbb{R}^n$.

Definition 4.

A finite sequence of vectors $\{v_1, v_2, \ldots, v_k\}$ *in* $\mathbb{R}^n$ *is called a* **Parseval sequence** *if it satisfies an identity like Parseval's identity. Specifically, for every* $x \in \mathbb{R}^n$,

$$\|x\|^2 = \sum_{i=1}^{k} |\langle x, v_i \rangle|^2.$$

1. Orthonormal bases for $\mathbb{R}^n$ are naturally examples of Parseval sequences, since they satisfy Parseval's identity, but there are other examples as well.
2. Notice that for a sequence to be a Parseval sequence for $\mathbb{R}^n$, it must have at least n elements.

Proposition 5.

Every Parseval sequence for $\mathbb{R}^n$ *is necessarily an* $\mathbb{R}^n$*-frame.*

We will henceforth call such sequences Parseval frames rather than Parseval sequences.

Lemma 6.

A Parseval frame $\{v_1, v_2, \dots, v_k\}$ *for* $\mathbb{R}^n$ *has an extension to an orthonormal basis of* $\mathbb{R}^k$.

The following is an immediate consequence of the above Lemma.

Lemma 7.

A finite sequence of vectors $\{v_1, v_2, \dots, v_k\}$ *in* $\mathbb{R}^n$ *is a Parseval sequence if and only if the rows of the matrix*

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}$$

are orthonormal vectors in $\mathbb{R}^k$.

Consider the following example in $\mathbb{R}^2$.

Example 8.

Let $\mathcal{H} = \mathbb{R}^2$, and let $\{x_1, x_2, x_3\}$ be the vectors

$$\left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}.$$

This collection of vectors is neither linearly independent nor an orthonormal set, but it shares some properties, including Parseval's identity, with an orthonormal basis.

Let $\mathcal{H}$ be a Hilbert space with finite dimension n. Our primary examples of such spaces will be the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$.

Definition 9.

A sequence of vectors $\{x_i\}_{i=1}^k \subset \mathcal{H}$ *is called a* **Parseval frame** *for* $\mathcal{H}$ *if for every* $x \in \mathcal{H}$*:*

$$\|x\|^2 = \sum_{i=1}^{k} |\langle x, x_i \rangle|^2. \tag{1}$$

Parseval frames are so named because Equation (1) looks like Parseval's identity for orthonormal bases. As we observed, every orthonormal basis is a Parseval frame but there do exist Parseval frames which are not orthonormal bases.

Equation (1) gives the definition of a Parseval frame, but it turns out we could also have used, as the defining property, the fact that every vector x can be reconstructed using the inner products $\langle x, x_i \rangle$ for coefficients. We now prove that the two notions are equivalent.

Proposition 10.

A collection of vectors $\{x_i\}_{i=1}^{k}$ is a Parseval frame for a Hilbert space $\mathcal{H}$ if and only if the following formula holds for every x in $\mathcal{H}$:

$$x = \sum_{i=1}^{k} \langle x, x_i \rangle x_i. \tag{2}$$

Equation (2) is called the **reconstruction formula for a Parseval frame***.*

Let Θ be the linear operator from $\mathcal{H}$ to $\mathbb{C}^k$ given by

$$\Theta x = \begin{bmatrix} \langle x, x_1 \rangle \\ \langle x, x_2 \rangle \\ \vdots \\ \langle x, x_k \rangle \end{bmatrix} = \sum_{i=1}^{k} \langle x, x_i \rangle e_i,$$

where $\{e_i\}_{i=1}^{k}$ is the standard orthonormal basis for $\mathbb{C}^k$.

This operator, called the **analysis operator**.

Remark 11.

1. *If* $\{x_i\}_{i=1}^k$ *is a Parseval frame, we can deduce that* $\|x_i\| \leq 1$ *for each* i.
2. *Let* $\{x_i\}_{i=1}^k$ *be a Parseval frame. If* $x_i \neq 0$, *then* $\|x_i\| = 1$ *if and only if* x_i *is orthogonal to each* x_j *when* $j \neq i$.

These two remarks yield the following observation, which illustrates an important difference between an orthonormal basis and a general Parseval frame.

Proposition 12.

Let $\{x_i\}_{i=1}^k$ *be a Parseval frame for a Hilbert space* $\mathcal{H}$.*Then it is an orthonormal basis if and only if each* x_i *is a unit vector.*

We also have the following formula on the dimension of the Hilbert space $\mathcal{H}$.

Proposition 13.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. If $\{x_i\}_{i=1}^k$ is a Parseval frame for $\mathcal{H}$, then

$$\dim \mathcal{H} = \sum_{i=1}^k \|x_i\|^2.$$

From the above result, we also have a new formula for the trace of a linear operator using a Parseval frame.

Corollary 14 (Trace Formula).

Let $\{x_i\}_{i=1}^k$ be a Parseval frame for $\mathcal{H}$, and let A be a linear operator on $\mathcal{H}$. Then $tr(A) = \sum_{i=1}^k \langle Ax_i, x_i \rangle$.

General Frames and the Canonical Reconstruction Formula.

Parseval frames are a special example of sets of vectors called frames. We introduced them first to emphasize the similarity with orthonormal bases. Frames, too, are related to bases, but they are more general than the Parseval frames. The following definition is the general frame analog of Definition 9.

Definition 15.

A **frame** *for a Hilbert space* $\mathcal{H}$ *is a sequence of vectors* $\{x_i\} \subset \mathcal{H}$ *for which there exist constants* $0 < A \leq B < \infty$ *such that, for every* $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B\|x\|^2. \tag{3}$$

The sum in Equation (3) will be considered to be a finite sum, since we are considering in the lecture finite frames for finite-dimensional Hilbert spaces.

We introduced frames for $\mathbb{R}^n$ and discovered that they are exactly the spanning sets of $\mathbb{R}^n$. The same result holds for general finite-dimensional Hilbert spaces.

Proposition 16.

Suppose that $\mathcal{H}$ is a finite-dimensional Hilbert space and $\{x_i\}_{i=1}^k$ is a finite collection of vectors from $\mathcal{H}$. Then the following are equivalent:

(i) $\{x_i\}_{i=1}^k$ *is a frame for* $\mathcal{H}$.

(ii) *span* $\{x_i\}_{i=1}^k = \mathcal{H}$.

The hypothesis that "**the Hilbert space is finite-dimensional**" cannot be dropped. The statement is not valid in infinite dimensions.

For general frames, we can also work out a reconstruction formula which resembles the reconstruction formula (2) for Parseval frames.

Proposition 17.

Let $\{x_i\}_{i=1}^k$ be a frame for $\mathcal{H}$. Then there exists a frame $\{y_i\}_{i=1}^k$ such that every $x \in \mathcal{H}$ can be reconstructed with the formaula.

$$x = \sum_{i=1}^{k} \langle x, y_i \rangle x_i = \sum_{i=1}^{k} \langle x, x_i \rangle y_i. \tag{4}$$

Such a frame $\{y_i\}_{i=1}^k$ is called a **dual frame** *to $\{x_i\}_{i=1}^k$.*

General Frames and the Canonical Reconstruction Formula.

Let Θ be the linear map from $\mathcal{H}$ to $\mathbb{C}^k$ defined by:

$$\Theta x = \begin{bmatrix} \langle x, x_1 \rangle \\ \langle x, x_2 \rangle \\ \vdots \\ \langle x, x_k \rangle \end{bmatrix}$$

We show that Θ is injective (one-to-one) by showing that the kernel is $\{0\}$. Let $\Theta x = 0$, which gives $\langle x, x_i \rangle = 0$ for $i = 1, 2, \ldots, k$. Since $\{x_i\}_{i=1}^k$ is a spanning set for $\mathcal{H}$, we can conclude that $x = 0$. Hence Θ is a bijection onto its range.

If we let Θ^* be the adjoint matrix of Θ, we may also conclude that the operator $S = \Theta^* \Theta : \mathcal{H} \to \mathcal{H}$ is invertible.

We call S the **frame operator** for the collection $\{x_i\}_{i=1}^k$.

Notation

As a remark on the notation here, notice that although we are using set or sequence notation for frames, neither is exactly the right notion.

For example, we need the vector e_1 to be in the collection twice in order for $\{e_1, e_2, e_1, -e_2\}$ to be a tight frame. To allow for repeated vectors, frames are often regarded as finite sequences rather than sets.

But the sequence notation is also an imperfect model, since the ordering it imposes would require us to regard a rearrangement of the same set of vectors as a different frame.

We will generally use the word collection of vectors, by which we mean to consider rearrangements to be equivalent but to allow elements to appear more than once.

In a finite-dimensional Hilbert space, a sequence of vectors is a spanning set if and only if it is a frame. Therefore, it is clear that if $\mathcal{H}$ has dimension n, any frame for $\mathcal{H}$ must have at least n vectors.

When a frame has more vectors than the dimension of the Hilbert space, we say it has an **increased redundancy**.

Definition 18.

Let $\mathcal{H}$ have dimension n and let $\{x_i\}_{i=1}^{k}$ be a frame for $\mathcal{H}$ having k elements. The redundancy of the frame is the quantity $\frac{k}{n}$, which must be greater than or equal to 1.

General Frames and the Canonical Reconstruction Formula.

The frame in Example 8, then, must have redundancy $\frac{3}{2}$. The redundancy of a frame gives a way to describe the size of the frame in comparison to the vector space it spans. In an application where the representation of a signal or image might be subjected to noise or erasures, redundancy helps to reduce the losses and errors that can occur.

Think of a vector x in $\mathcal{H}$ as a sentence spoken into a cell phone. The voice message is broken down into a series of coefficients which are digitized, transmitted, and read by a receiver which knows how to transform the coefficients back into an audible sentence.

During the transmission, however, some of those coefficients may be scrambled up or erased. If there were some extra coefficients used, we have a better chance of understanding the message on the other end, even if it is not perfectly identical to the message that was sent.

General Frames and the Canonical Reconstruction Formula.

Another advantage to redundancy is the variety of frames that exist. Frames are used in a wide variety of applications, each having unique constraints. Orthonormal bases are very restrictive. The elements in an orthonormal basis must all be orthogonal, there can only be exactly as many elements as the dimension of the space, and they must all have unit norm.

Frames can be structured to adjust the weight on each component by having vectors with varying norms. Frames exist which pay special attention to some parts of a signal by grouping more vectors in these areas. This is accomplished by constructing a frame with varied spacing (measured by the inner products) between vectors.

General Frames and the Canonical Reconstruction Formula.

The property that is present in orthonormal bases and lost for frames with redundancy greater than 1 is the uniqueness of the coefficients. In certain applications, however, uniqueness may be less important than the benefits from redundancy and variety of structures.

In an infinite-dimensional space, the analog of a spanning set is a complete sequence, and it turns out to be untrue that every complete sequence is a frame. There is a rich theory of frames in infinite dimensions.

Frames and Matrices

The map Θ was introduced which led to the reconstruction formula in Equation (4). We now see a more detailed investigation of this operator. For convenience we always assume that every vector $x \in \mathcal{H}$ has a vector form with respect to a fixed basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathcal{H}$.

Definition 19.

Let $\{x_i\}_{i=1}^k \subset \mathcal{H}$. The matrix (operator) $\Theta : \mathcal{H} \to \mathbb{C}^k$ defined by

$$\Theta x = \begin{bmatrix} \langle x, x_1 \rangle \\ \vdots \\ \langle x, x_k \rangle \end{bmatrix} = \sum_{i=1}^{k} \langle x, x_i \rangle e_i$$

is called the **analysis matrix (or operator)** *of $\{x_i\}_{i=1}^k$, where $\{e_i\}_{i=1}^k$ is the standard orthonormal basis for $\mathbb{C}^k$.*

Analysis operator

We now give a more detailed investigation of the analysis operator.

Definition 20.

The adjoint Θ^* *of the analysis operator, which will map* $\mathbb{C}^k$ *to* $\mathcal{H}$*, is called the* **synthesis operator***. The operator* $S = \Theta^*\Theta : \mathcal{H} \to \mathcal{H}$ *is called the* **frame operator***.*

We can prove that S *is a positive self-adjoint operator. Note that we can define these operators* **for any sequence of vectors** $\{x_i\}_{i=1}^k$ *from* $\mathcal{H}$*, although* $\{x_i\}_{i=1}^k$ *may not be a frame.*

The definition of a Parseval frame and Proposition 10 combine to prove that the frame operator is exactly the identity operator for $\mathcal{H}$; i.e., $Sx = x$, for all $x \in \mathcal{H}$, if and only if $\{x_i\}_{i=1}^k$ is a Parseval frame for $\mathcal{H}$. It is an immediate corollary that $\{x_i\}_{i=1}^k$ is a tight frame if and only if the frame operator for $\{x_i\}_{i=1}^k$ is a scalar multiple of the identity operator.

Analysis operator

Recall the handy equation from Proposition 17, using the norm in $\mathbb{C}^k$:

$$\|Tx\|^2 = \sum_{i=1}^{k} |\langle x, x_i\rangle|^2. \tag{5}$$

This will lead us to the following equivalent ways to describe a frame for a finite-dimensional Hilbert space.

Lemma 21.

Let $\mathcal{H}$ be a Hilbert space and let $\{x_i\}_{i=1}^k \subset \mathcal{H}$. Let Θ be the analysis operator for $\{x_i\}_{i=1}^k$. Then the following are equivalent:

(i) *$\{x_i\}_{i=1}^k$ is a frame for $\mathcal{H}$.*

(ii) *Θ is a one to-one operator from $\mathcal{H}$ to $\mathbb{C}^k$.*

(iii) *The frame operator $S = \Theta^*\Theta$ is an invertible operator on $\mathcal{H}$.*

Moreover, if $\{x_i\}_{i=1}^k$ is a frame for $\mathcal{H}$, then $A = \frac{1}{\|\Theta^{-1}\|^2}$ and $B = \|\Theta\|^2$ are frame bounds, where $\Theta^{-1} : \Theta(\mathcal{H}) \to \mathcal{H}$ is the inverse of Θ.

Analysis operator

Now we view Θ as a $k \times n$ matrix with respect to the standard orthonormal basis $\{e_i\}_{i=1}^k$ for $\mathbb{C}^k$ and a fixed orthonormal basis $\{u_1, u_2, \dots, u_n\}$ for $\mathcal{H}$.

Write $\Theta = [\Theta_{ij}]_{k \times n}$.

Then we have

$$\Theta_{ij} = \langle \Theta u_j, e_i \rangle = \langle u_j, \Theta^* e_i \rangle = \langle u_j, x_i \rangle = \overline{\langle x_i, u_j \rangle}.$$

The rows of the matrix Θ are $x_i^* = \bar{x}_i^T$ if we view each vector x_i as a column vector in $\mathcal{H}$:

$$\Theta = \begin{bmatrix} \leftarrow x_1^* \rightarrow \\ \leftarrow x_2^* \rightarrow \\ \vdots \\ \leftarrow x_k^* \rightarrow \end{bmatrix}.$$

Analysis operator

Similarly, the synthesis operator Θ^* is an $n \times k$ matrix with the vectors x_i as its columns:

$$\Theta^* = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \dots & x_k \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Using the matrix representations of the analysis and synthesis operators, we find the following properties:

Proposition 22.

Let $T = [x_1 \quad x_2 \quad \dots \quad x_k]$ be an $n \times k$ matrix with x_i being the column vectors of T.

(i) *$\{x_1, x_2, dots, x_k\}$ is a frame for $\mathbb{C}^n$ if and only if T has rank n.*

(ii) *$\{x_1, x_2, \dots, x_k\}$ is a tight frame for $\mathbb{C}^n$ if and only if the set of row vectors of T is a pairwise orthogonal collection of vectors all having the same norm. In particular, $\{x_i\}_{i=1}^k$ is a Parseval frame for $\mathbb{C}^n$ if and only if the set of row vectors of T is an orthonormal set.*

Frames and Matrices

Motivated by the above result, we give the following definitions:

Definition 23.

A matrix T of size $n \times k$ is called a **frame matrix** *if it has rank n. T is called a* **Parseval frame matrix** *if $TT^* = I_{n\times n}$. It is called a* **tight frame matrix** *if $TT^* = \lambda I_{n\times n}$ for some $\lambda > 0$. If T is a tight frame matrix and, in addition, all of the columns of T have the same norm, then T is called an* **equi-norm tight frame matrix**.

Proposition 24.

Let $T = [x_1 \quad x_2 \quad \dots \quad x_k]$ be an $n \times k$ matrix and let λ_{min} and λ_{max} respectively be the minimal and maximal eigenvalues of TT^*. Then following are equivalent :

(i) $\lambda_{min} > 0$;

(ii) $\{x_i\}_{i=1}^k$ *is a frame for* $\mathbb{C}^n$.

Moreover, when the above statements hold, the optimal lower and upper frame bounds for $\{x_i\}_{i=1}^k$ are λ_{min} and λ_{max}, respectively.

Frames and Matrices

If $\{x_i\}_{i=1}^k$ is a frame with optimal frame bounds A and B, then

$$\frac{B}{A} = \frac{\lambda_{max}}{\lambda_{min}} = c(S)$$

is the **condition number** of the frame operator S, which is an important quantity in the numerical computation of S^{-1}, and will be discussed later in more detail.

Given the analysis operator Θ for a sequence $\{x_i\}_{i=1}^k$, we have been describing the importance of the frame operator$S = \Theta^*\Theta$.

Composing the analysis and synthesis operators in the other order $G = \Theta\Theta^*$ gives an operator called the **Grammian operator** on the Hilbert space $\mathbb{C}^k$. The Grammian also plays an important role in frame theory.

Frames and Matrices

The entries of the Grammian matrix are the inner products between the frame elements:

$$G = \Theta\Theta^* = \begin{bmatrix} \langle x_1, x_1\rangle & \langle x_2, x_1\rangle & \dots & \langle x_k, x_1\rangle \\ \langle x_1, x_2\rangle & \langle x_2, x_2\rangle & \dots & \langle x_k, x_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, x_k\rangle & \langle x_2, x_k\rangle & \dots & \langle x_k, x_k\rangle \end{bmatrix}.$$

The diagonal entries of the Grammian are $\|x_i\|^2, i = 1, 2, \dots, k$, the squares of the norms of the frame elements.

Proposition 25.

A collection of vectors $\{x_i\}_{i=1}^k \subset \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of dimension n, is a Parseval frame for $\mathcal{H}$ if and only if the associated Grammian operator G is an orthogonal projection of rank n.

Similarity and Unitary Equivalence of Frames

There are several commonly used notions of equivalence among frames. In other words, there are frames which, although they are technically different, are considered to be the "same". Below are some natural notions of equivalent frames in $\mathbb{R}^n$.

1. Frames which contain the same vectors, but given in a different order.
2. Frames which only differ in that some of the vectors x_i have been replaced with the additive inverse $-x_i$.
3. Frames which are rotations of each other, in the sense that a fixed rotation is applied to each vector of one frame to create the other frame.

Similarity and Unitary Equivalence of Frames

Lemma 26.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Given $\{x_i\}_{i=1}^k \subset \mathcal{H}$ and T any invertible (bijective) operator from $\mathcal{H}$ to $\mathcal{K}$, $\{x_i\}_{i=1}^k$ is a frame for $\mathcal{H}$ if and only if $\{Tx_i\}_{i=1}^k$ is a frame for $\mathcal{K}$.

This lemma motivates our definition of two types of equivalence of frames:

Definition 27.

Two frames $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively are said to be **similar** *if there exists an invertible operator $T : \mathcal{H} \to \mathcal{K}$ such that $T_{x_i} = y_i$ for $i = 1, 2, \ldots, k$. The frames are called* **unitarily equivalent** *if we require T to be a unitary operator from $\mathcal{H}$ to $\mathcal{K}$.*

We remark that similarity is an equivalence relation which is order dependent. For example, if $\{e_i\}_{i=1}^n$ is an orthonormal basis for $\mathcal{H}$, then $\{0, e_1, \ldots, e_n\}$ and $\{e_1, 0, e_2, \ldots, e_n\}$ are two nonsimilar frames, although they are the same set.

Similarity and Unitary Equivalence of Frames

The following result tells us the rather interesting fact that every frame is similar to a Parseval frame.

Proposition 28.

Let $\{x_i\}_{i=1}^k$ be a frame for $\mathcal{H}$ with frame operator S. Then $\{S^{-\frac{1}{2}}x_i\}_{i=1}^k$ is a Parseval frame for $\mathcal{H}$.

It is possible, though, for a frame to be similar to more than one Parseval frame. The following lemma describes similarity between two different Parseval frames.

Lemma 29.

If two Parseval frames are similar, then they must also be unitarily equivalent.

Similarity and Unitary Equivalence of Frames

We point out that if a frame $\{x_i\}_{i=1}^k$ is unitarily equivalent to a Parseval frame, then $\{x_i\}_{i=1}^k$ is also a Parseval frame. The next result gives us a characterization of similar frames:

Theorem 30.

Let $\{x_i\}_{i=1}^k$ and $\{y_i\}_{i=1}^k$ be two frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. Then they are similar if and only if their analysis operators have the same range.

Corollary 31.

A frame $\{x_i\}_{i=1}^k$ for a finite-dimensional Hilbert space $\mathcal{H}$ is also a basis if and only if the range of its analysis operator is the whole space $\mathbb{C}^k$.

Similarity and Unitary Equivalence of Frames

Theorem 30 induces a classification of all the frames having k vectors, according to the range of their analysis operators.

Corollary 32.

Let $\mathbb{J} = \{1, 2, \ldots, k\}$. Then the set of the equivalence classes (by similarity) of all the frames indexed by $\mathbb{J}$ is in one-to-one correspondence with the set of all subspaces of $\mathbb{C}^k$.

Numerical Algorithms

Let $\{x_i\}_{i=1}^k$ be a frame for a Hilbert space $\mathcal{H}$ with frame operator S. Suppose that the frame coefficients $\{\langle x, x_i \rangle\}_{i=1}^k$ are known. We want to reconstruct x by using these coefficients and the frame vectors $\{x_i\}_{i=1}^k$. Recall that we can do this by computing the canonical dual frame $\{S^{-1}x_i\}_{i=1}^k$ and using the reconstruction formula

$$x = \sum_{i=1}^{k} \langle x, x_i \rangle S^{-1} x_i.$$

In order to do this, however, we need to invert the frame operator S. The speed of convergence in the numerical procedure of finding S^{-1} depends heavily on the condition number of S. (Recall that the condition number is the ratio between the optimal upper frame bound and the optimal lower frame bound.) Therefore, inverting the frame operator by computer could be unacceptably time consuming if the dimension of $\mathcal{H}$ and the condition number are large.

Numerical Algorithms

An alternative to exactly reconstructing x is to use an algorithm which will produce increasingly accurate approximations of $x \in \mathcal{H}$ using the frame vectors and coefficients.

We now see three algorithms: *the frame algorithm, the Chebyshev algorithm, and the conjugate gradient algorithm*.

The frame algorithm is the simplest. However, the other two were showed by Grochenig to provide faster convergence when the condition number of S is very large.

The frame algorithm

Let $\{x_i\}_{i=1}^n$ be a frame for $\mathcal{H}$ with frame bounds A and B. Given $x \in \mathcal{H}$, we describe the following recursive algorithm:

$$u_0 = 0,$$
$$u_k = u_{k-1} + \frac{2}{A+B} S(x - u_{k-1}), \quad k \geq 1.$$

Note that the values

$$S(x - u_{k-1}) = \sum_{i=1}^{n} (\langle x, x_i \rangle + \langle u_{k-1}, x_i \rangle) x_k$$

can be easily calculated since $\{\langle x, x_i \rangle\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$ are already given. The convergence speed of this algorithm is given by

$$\|x - u_i\| \leq \left(\frac{B-A}{A+B} \right)^i \|x\|.$$

The Chebyshev algorithm

Let $\{x_i\}_{i=1}^n$ be a frame for $\mathcal{H}$ with frame bounds A and B. Given $x \in \mathcal{H}$, define

$$\rho = \frac{B-A}{B+A}, \quad \sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}.$$

Define the sequence $\{u_k\}_{k=0}^{\infty}$ in $\mathcal{H}$ and corresponding numbers $\{\lambda_k\}_{k=1}^{\infty}$ by

$$u_0 = 0, \quad u_1 = \frac{2}{B+A} Sx, \quad \lambda_1 = 2,$$

and for $k \geq 2$, $\lambda_k = \frac{1}{1-\frac{\rho^2}{4}\lambda_{k-1}}$ and
$u_k = \lambda_k(u_{k-1} - u_{k-2} + \frac{2}{B+A}S(f - u_{k-1})) + u_{k-2}$.

Then the sequence $\{u_k\}$ converges to x in $\mathcal{H}$, and the speed of convergence for the Chebyshev algorithm is:

$$\|x - u_k\| \leq \frac{2\sigma^k}{1+\sigma^{2k}} \|x\|.$$

The conjugate gradient algorithm

Unlike the previous two algorithms, the conjugate gradient algorithm works without the knowledge of the frame bounds (but the speed of convergence does depend on the frame bounds). Let $\{x_i\}_{i=1}^n$ be a frame for $\mathcal{H}$ with frame bounds A and B. Let $x \in \mathcal{H}$ be a nonzero vector. Define three sequences $\{u_k\}_{k=0}^\infty$, $\{r_k\}_{k=0}^\infty$, and $\{p_k\}_{k=1}^\infty$ in $\mathcal{H}$, and corresponding scalars $\{\lambda_k\}_{k=-1}^\infty$ by

$$u_0 = 0, \quad r_0 = p_0 = Sx, \quad p_{-1} = 0$$

and for $k \geq 0$,

$$\begin{aligned}
\lambda_k &= \frac{\langle r_k, p_k \rangle}{\langle p_k, Sp_k \rangle}, \\
u_{k+1} &= u_k + \lambda_k p_k, \\
r_{k+1} &= r_k - \lambda_k S p_k, \\
p_{k+1} &= Sp_k - \frac{\langle Sp_k, Sp_k \rangle}{\langle p_k, Sp_k \rangle} p_k - \frac{\langle Sp_k, Sp_{k-1} \rangle}{\langle p_{k-1}, Sp_{k-1} \rangle} p_{k-1}.
\end{aligned}$$

The sequence u_k converges in $\mathcal{H}$ to the vector x.

Exercises 33.

1. *Let* A *and* B *be* $n \times n$ *positive self-adjoint matrices such that for all* $x \in \mathbb{C}^n$, $x^*Ax = x^*Bx$. *Prove that* $A = B$. *Equivalently, prove that if* A, B *are positive operators on* $\mathcal{H}$ *such that* $\langle Ax, x \rangle = \langle Bx, x \rangle$, *for all* $x \in \mathcal{H}$, *then* $A = B$.
2. *Prove that if* $\mathcal{H}$ *is a finite-dimensional Hilbert space, then every finite sequence has an upper frame bound.*
3. *Prove that if* $\{x_i\}_{i=1}^k$ *is a tight frame of unit vectors for an n-dimensional Hilbert space, then the frame bound is* $A = \frac{k}{n}$. *More generally, if* $\{x_i\}_{i=1}^k$ *is a tight frame for* $\mathcal{H}$, *then the frame bound is given by:* $A = \frac{1}{n} \sum_{i=1}^k \|x_i\|^2$.
4. *Verify that the matrix of the Grammian operator is* $[G]_{i,j} = \langle x_j, x_i \rangle$.
5. *Prove that if* $\{x_i\}_{i=1}^k$ *is a frame with upper frame bound* B, *then* $\|x_i\|^2 \leq B$ *for all* $i = \{1, 2, \ldots, k\}$.

Exercises 34.

1. *Let* $\{x_i\}_{i=1}^k$ *be any sequence of vectors in* $\mathcal{H}$ *with* $k \geq n$ *and let* S *be the associated frame operator. (Note that a sequence need not be a frame to form the frame operator.) Let* $b_1 \leq b_2 \leq \cdots \leq b_n$ *be the eigenvalues of* S*. If* $b_1 > 0$*, prove that the sequence is a frame for* $\mathcal{H}$ *with optimal frame bounds* b_1 *and* b_n*. [Hint: Since* S *is a positive operator, the Spectral Theorem implies that there is an orthonormal basis* $\{u_i\}_{i=1}^n$ *of eigenvectors of* S*.]*
2. *Let* $\{x_i\}_{i=1}^k$ *be a frame with lower and upper frame bounds* A *and* B *respectively, and let* S *be the associated frame operator. Prove that* $\{S^{-1}x_i\}_{i=1}^k$ *is also a frame with lower frame bound* $\frac{1}{B}$ *and upper frame bound* $\frac{1}{A}$*.*

Exercises 35.

1. *Let G be the Grammian operator for a frame $\{x_i\}_{i=1}^k$ in $\mathbb{R}^n$. Prove that G is invertible if and only if $k = n$.*
2. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. For $A \in \mathcal{B}(\mathcal{H})$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2 = [tr(A^*A)]^{\frac{1}{2}}$.*
   (i) *Prove that $\|\cdot\|_2$ is, in fact, a norm on the vector space $\mathcal{B}(\mathcal{H})$.*
   (ii) *Prove that $\|\cdot\|_2$ is submultiplicative, which means that for all $A, B \in \mathcal{B}(\mathcal{H})$, $\|AB\|_2 \leq \|A\|_2\|B\|_2$.*
   (iii) *More generally, for any fixed p, $1 \leq p \leq \infty$, we can define a norm on $\mathcal{B}(\mathcal{H})$ called the* **Shatten p-norm**, *given by $||A||_p = [tr(|A|^p)]^{\frac{1}{p}}$. Prove that $\|\cdot\|_p$ is, in fact, a norm on $\mathcal{B}(\mathcal{H})$.*

Frames in $\mathbb{R}^2$

This chapter develops a geometric description of tight frames in the plane $\mathbb{R}^2$. While this characterization of tight frames does not extend to higher-dimensional spaces, we can use it to develop some intuition about the types of tight frames that exist. We will discover some properties of $\mathbb{R}^2$ frames, and these properties will generate a variety of questions, some still unanswered, about frames in $\mathbb{R}^n$.

A collection of vectors $\{x_i\}_{i=1}^k$, where $x_i = \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \end{bmatrix}$, is a tight frame with frame bound A if and only if the frame operator S is equal to A times the identity operator. The frame operator for a collection of vectors $\left\{ \begin{bmatrix} a_i \cos\theta_i \\ a_i \sin\theta_i \end{bmatrix} \right\}_{i=1}^k$ in $\mathbb{R}^2$ will be the 2×2 matrix:

$$S = \Theta^*\Theta = \begin{bmatrix} \sum_{i=1}^k a_i^2 \cos^2\theta_i & \sum_{i=1}^k a_i^2 \cos\theta_i \sin\theta_i \\ \sum_{i=1}^k a_i^2 \cos\theta_i \sin\theta_i & \sum_{i=1}^k a_i^2 \sin^2\theta_i \end{bmatrix}.$$

Frames in $\mathbb{R}^2$

Using trigonometric double-angle formulas, we see that the frame operator S is a scalar multiple of the identity operator if and only if the following vector equation holds:

$$\sum_{i=1}^{k} \begin{bmatrix} a_i^2 \cos 2\theta_i \\ a_i^2 \sin 2\theta_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

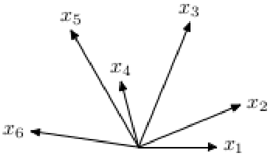
This establishes an exact correspondence between tight frames for $\mathbb{R}^2$ and collections of vectors obtained from the tight frame vectors by squaring the length and doubling the angle of each vector; the original collection is a tight frame exactly when the new collection of vectors sums to zero.

If $x = \begin{bmatrix} a\cos\theta \\ a\sin\theta \end{bmatrix}$ is a vector in $\mathbb{R}^2$, let $\tilde{x} = \begin{bmatrix} a^2\cos 2\theta \\ a^2\sin 2\theta \end{bmatrix}$ be this associated vector which we will call the **diagram vector**, because we will be drawing diagrams in which we discuss the sum of such vectors.

Frames in $\mathbb{R}^2$

Lemma 36.

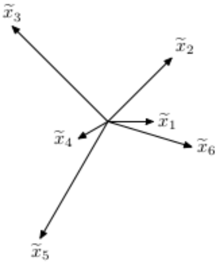
A collection $\{x_i\}_{i=1}^k$ of vectors in $\mathbb{R}^2$, where $k \geq 2$, is a tight frame for $\mathbb{R}^2$ iff the diagram vectors $\{\tilde{x}_i\}_{i=1}^k$ sum to zero in $\mathbb{R}^2$.

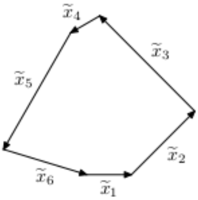


These vectors form a tight frame for $\mathbb{R}^2$.

Frames in $\mathbb{R}^2$

The diagram of the sum of vectors $\{\tilde{x}_i\}_{i=1}^k$ provides a visual representation of the tight frames in $\mathbb{R}^2$, and provides very intuitive answers to questions we may ask about frames.





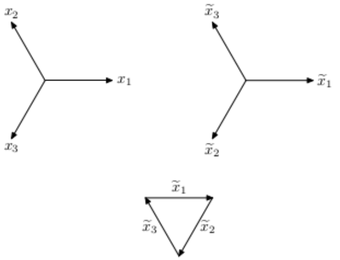
The diagram vectors here are shown first in standard position, and then placed tip-to-tail to demonstrate that they sum to zero.

Frames in $\mathbb{R}^2$

One example of a unit tight frame with three vectors in $\mathbb{R}^2$ can be formed by rescaling the vectors from Example 8 to unit vectors :

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}.$$

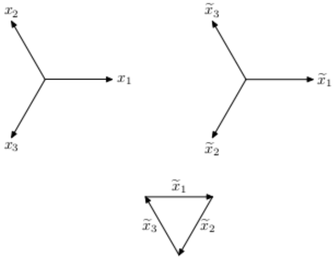
These vectors and their diagram vectors are shown in the following figure, and we see that indeed the diagram vectors sum to zero.



If we want to consider all the unit tight frames having three vectors, we can consider the possible diagrams of unit vectors which can sum to zero. Since we are restricted to unit vectors in this case, we realize that the only possible diagrams will be equilateral triangles, which fixes the angles between the diagram vectors (see the Figure).

Equivalence of Frames

Suppose that a collection of vectors $\{x_i\}_{i=1}^{k}$ is a frame or a tight frame. There are certain other collections which we would consider to be in some sense the same frame.



Here is the unit frame of three vectors given by scaling the vectors in Example 8, followed by the associated diagram vectors. In this special example, the set of diagram vectors is simply a rearrangement of the original set of vectors. Observe that the diagram vectors sum to zero, which proves that this frame is a tight frame for $\mathbb{R}^2$.

Equivalence of Frames

We can carefully define an equivalence relation among such frames, so that we can talk about these frames collectively as a single frame.

We will describe a new equivalence relation in which frames containing the same vectors in different orders are equivalent. We will also consider collections for which some vectors x_i are replaced with $-x_i$ and collections which are a uniform rotation from the original one to be equivalent to the original one.

This equivalence we are describing here is not the one most commonly called **frame equivalence**, so we will use the term PRR-equivalence (permutation-reflection-rotation equivalence) to avoid any confusion.

PRR-equivalence will equate two frames which contain the same elements, only permuted. The order in which the vectors are listed may change the analysis, synthesis, and frame operators.

Equivalence of Frames

In the same way, if we replace x_i with $-x_i$, we can verify that the frame bound and frame operator do not change. So the act of replacing a subset of frame vectors with their additive inverse should not change the frame, in our description of different frames. If each vector is rotated by a fixed angle θ, the resulting collection is also a frame, with the same properties as the original frame.

The three equivalences - permutation, reflection, rotation - are naturally seen to give rise to diagrams that are also, in some sense, the same. If the order of the vectors in an $\mathbb{R}^2$ frame is altered, the diagram vectors are the same, but also permuted in order. Because there are a finite number of diagram vectors, the sum of these diagram vectors is certainly unchanged by the rearrangement. If the vectors are placed tip-to-tail, the resulting plane figure will look different, but will begin and end at the same points as the original figure. If the frame was originally a tight frame, both diagrams will be closed polygons.

Equivalence of Frames

If each vector in an $\mathbb{R}^2$ frame is rotated by an angle θ, the resulting diagram vectors will each be rotated by 2θ, and the tip-to-tail figure will just be a rotation of the original figure.

Next, observe that the additive inverse $-x_i$ of a vector x_i in an $\mathbb{R}^2$ frame has exactly the same diagram vector as x_i, since $2(\theta + \pi) = 2\theta + 2\pi$. Therefore, the tip-to-tail diagram is completely unchanged when some of the frame vectors are replaced with their additive inverses.

With the definition of PRR-equivalence at hand, we can now classify all of the unit tight frames for $\mathbb{R}^2$ which have exactly three vectors. Since the only possible tip-to-tail diagram of three unit vectors which sum to zero forms an equilateral triangle, every tight frame must have this diagram.

Equivalence of Frames

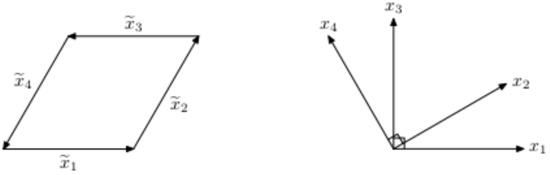
Therefore, there is only one PRR-equivalence class of unit tight frames for $\mathbb{R}^2$ with three vectors, and that equivalence class can be represented with the frame given in Example 8.

The diagram vectors must be equally spaced at angles of $\frac{2\pi}{3}$ when in standard position so that they form an equilateral triangle when placed tip-to-tail.

Suppose the angles of $\{\tilde{x}_i\}_{i=1}^3$ are $0^\circ, 120^\circ, 240^\circ$, then the angles of $\{x_i\}_{i=1}^3$ are $0^\circ, 60^\circ, 120^\circ$ respectively. This sequence $\{x_1, x_2, x_3\}$ does not have any orthonormal basis of $\mathbb{R}^2$.

Unit Tight Frames with Four Vectors

The diagram vectors also allow us to characterize the unit tight frames for $\mathbb{R}^2$ having exactly four vectors. The tip-to-tail diagram of such a frame would be a quadrilateral with equal sides (a rhombus), which must necessarily be a parallelogram as well. In other words, the diagram consists of two pairs of vectors which are additive inverses of each other. This implies that the original frame must contain two pairs of orthonormal vectors.



The diagram for a 4-vector unit tight frame for $\mathbb{R}^2$ must be a parallelogram, so the original frame must have two pairs of orthogonal vectors.

Unit Tight Frames with Four Vectors

This discussion leads to the following proposition.

Proposition 37.

Every unit tight frame for $\mathbb{R}^2$ having exactly four vectors must be the union of two orthonormal bases for $\mathbb{R}^2$.

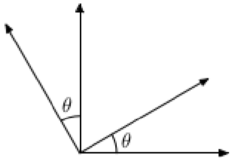
From this proposition, the set of all nonPRR-equivalent unit tight frames with four vectors can be characterized by the angle between nonorthogonal vectors.

In other words, there is a one-parameter family of unit tight frames with four vectors in $\mathbb{R}^2$, where the parameter is the angle between two nonorthogonal vectors.

Unit Tight Frames with Four Vectors

Proposition 38.

Given any θ, $0 \leq \theta < \frac{\pi}{2}$, there exists a unit tight frame F_θ with four vectors for $\mathbb{R}^2$. If $\theta_1 \neq \theta_2$, then F_{θ_1} is not PRR-equivalent to F_{θ_2}.



The angle θ characterizes all the nonPRR-equivalent unit tight frames for $\mathbb{R}^2$ with four vectors.

Unit Tight Frames with Four Vectors

Proposition 39.

Given $k \geq 3$, *the sequence* $\mathcal{F} = \left\{ \begin{bmatrix} \cos\frac{2\pi j}{k} \\ \sin\frac{2\pi j}{k} \end{bmatrix} \right\}_{j=0}^{k-1}$ *is a tight frame for* $\mathbb{R}^2$.

Another property we could examine is whether a tight frame for $\mathbb{R}^2$ can be decomposed into two other tight frames. For example, we showed above that a unit tight frame with four vectors for $\mathbb{R}^2$ is always a union of two orthonormal bases, but clearly a unit tight frame with three vectors does not contain enough vectors to be decomposed into two spanning sets for $\mathbb{R}^2$.

Proposition 40.

(i) *For every* $k \geq 2$, *there exists a unit tight frame for* $\mathbb{R}^2$ *having* k *vectors.*

(ii) *For* $k \neq 4$, *there exists a unit tight frame for* $\mathbb{R}^2$ *which is not the union of two or more existing tight frames for* $\mathbb{R}^2$.

Notion of diagram vectors was extended to $\mathbb{R}^n$ and $\mathbb{C}^n$.

Any vector f in $\mathbb{R}^2$ can be written as $f = \begin{bmatrix} f(1) \\ f(2) \end{bmatrix}$. The associated diagram vector $\tilde{f}$ is defined as $\tilde{f} = \begin{bmatrix} f^2(1) - f^2(2) \\ 2f(1)f(2) \end{bmatrix}$. Diagram vectors are used in the following characterization of tight frames.

Theorem 41.

Let $\{f_i\}_{i \in I}$ be a sequence of vectors in $\mathbb{R}^2$, not all of which are zero. Then $\{f_i\}_{i \in I}$ is a tight frame if and only if $\sum_{i \in I} \tilde{f}_i = 0$.

The diagram vector of a vector in $\mathbb{R}^2$ belongs to $\mathbb{R}^2$. The diagram vectors of a tight frame in $\mathbb{R}^2$ can be placed tip-to-tail to demonstrate that they sum to zero.

The notion of diagram vectors was extended to $\mathbb{R}^n$ and $\mathbb{C}^n$.

Notion of diagram vectors was extended to $\mathbb{R}^n$ and $\mathbb{C}^n$.

Definition 42.

For any vector $f = \begin{bmatrix} f(1) \\ \vdots \\ f(n) \end{bmatrix} \in \mathbb{R}^n$, *we define the diagram vector of* f, *denoted as* $\tilde{f}$, *by*

$$\tilde{f} = \frac{1}{\sqrt{n-1}} \begin{bmatrix} (f^2(i) - f^2(j))_{i<j} \\ (\sqrt{2n} f(i) f(j))_{i<j} \end{bmatrix} \in \mathbb{R}^{n(n-1)},$$

where the difference of squares $f^2(i) - f^2(j)$ *and the product* $f(i)f(j)$ *occur exactly once for* $i < j, i = 1, \ldots, n-1$.

Notion of diagram vectors was extended to $\mathbb{R}^n$ and $\mathbb{C}^n$.

Definition 43.

For any vector $f \in \mathbb{C}^n$, we define the diagram vector of f, denoted as $\tilde{f}$, by

$$\tilde{f} = \frac{1}{\sqrt{n-1}} \begin{bmatrix} (|f(i)|^2 - |f(j)|^2)_{i<j} \\ (\sqrt{n} f(i)\overline{f(j)})_{i<j} \\ (\sqrt{n}\overline{f(i)} f(j))_{i<j} \end{bmatrix} \in \mathbb{C}^{3n(n-1)/2},$$

where the difference of the form $|f(i)|^2 - |f(j)|^2$ occurs exactly once for $i < j, i = 1, \ldots, n-1$, and the product of the form $f(i)\overline{f(j)}$ occurs exactly once for $i \neq j$.

Notion of diagram vectors was extended to $\mathbb{R}^n$ and $\mathbb{C}^n$.

Using these definitions, Theorem 41 was extended to H_n (Hilbert space of dimension n).

Theorem 44.

Let $\{f_i\}_{i\in I}$ be a sequence of vectors in H_n, not all of which are zero. Then $\{f_i\}_{i\in I}$ is a tight frame if and only if $\sum_{i\in I} \tilde{f}_i = 0$. Moreover, for any $f, g \in H_n$ we have $(n-1)(\tilde{f} \cdot \tilde{g}) = n|(f \cdot g)|^2 - \|f\|^2\|g\|^2$.

From the above Theorem, it is immediate that if $\|f\| = 1$ then $\|\tilde{f}\| = 1$. Suppose $S^k := \{f \in \mathbb{R}^{k+1} : \|f\| = 1\}$ is the unit sphere in $\mathbb{R}^{k+1}$. We can define the diagram operator $D : S^{n-1} \to S^{n(n-1)-1}$ and note that D is not injective since f and $-f$ have the same diagram vectors. It can be shown that D is surjective if and only if $n = 2$.

Fundamental Inequality in $\mathbb{R}^2$

Proposition 45 (Fundamental Inequality in $\mathbb{R}^2$).

Let $\{a_i\}_{i=1}^k \subset \mathbb{R}^+$, with $k \geq 2$ and $a_1 \geq a_2 \geq \cdots \geq a_k > 0$. Then there exists a tight frame $\{x_i\}_{i=1}^k$ for $\mathbb{R}^2$ such that $\|x_i\|^2 = a_i$, $i = 1, 2, \ldots, k$ if and only if

$$a_1 \leq \sum_{i=2}^{k} a_i.$$

Exercises 46.

1. *Prove that for every* $k \geq 2$*, there exists a unit tight frame for* $\mathbb{R}^2$ *having* k *vectors.*
2. *Prove that for* $k \neq 4$*, there exists a unit tight frame for* $\mathbb{R}^2$ *which is not the union of two or more existing tight frames for* $\mathbb{R}^2$*.*
3. *Describe the set of all unit tight frames with exactly five vectors, up to PRR-equivalence.*
4. *Given a finite set of* k *unit vectors in* $\mathbb{R}^2$*, with* $k \geq 3$*, use diagram vectors to show that they can be scaled by positive constants to form a tight frame so long as their corresponding "double-angle vectors"* $\begin{bmatrix} \cos 2\theta_i \\ \sin 2\theta_i \end{bmatrix}$ *are not all contained in any open half-plane of* $\mathbb{R}^2$*.*

References

- Deguang Han, Keri Kornelson, David Larson and Eric Weber, *Frames for Undergraduates*, AMS, Student Mathematical Library, Volume 40, Rhode Island.
- Alice Z.-Y. Chan, Martin S. Copenhaver, Sivaram K. Narayan, Logan Stokols, Allison Theobold, *On Structural Decompositions of Finite Frames*, Advances in Computational Mathematics, Volume 42, 721-756 (2016).